Indian J. Phys. **70A** (3) 413–421 (1996)

IJP A
– an international journal

Eikonal approximation in light scattering

S.K. Sharma
S.N. Bose National Centre for Basic Sciences
Block JD, Sector III, Salt Lake, Calcutta-700091, INDIA

Abstract : The role that the eikonal approximation plays in the analysis of light scattering by soft particles has been reviewed briefly.

Key Words : Eikonal Approximation, Light Scattering.
PACS NOS : 42.10 Hc

1. Introduction

The eikonal approximation (EA) [1], sometimes also referred to as the high energy approximation, is known to be a good approximation for a long time for small angle elastic scattering of a plane wave by a potential V($\mathbf{r}$) in the domain $\frac{|V(\mathbf{r})|}{k^2} \ll 1$ and $ka \gg 1$. Here k is the wave number of the incident wave and a is the range of the potential.

By drawing an analogy between a potential V($\mathbf{r}$) and refractive index n($\mathbf{r}$) for a particle, the EA has been successfully adapted in light scattering problems by defining an effective potential $V(\mathbf{r}, k) = -k^2[n^2(\mathbf{r}) - 1]$. Here it has been applied to a variety of problems [2]. Our aim here is to review briefly the validity and applications of this approximation in this context.

2. The eikonal approximation in potential scattering

Consider the nonrelativistic scattering of a particle by a potential V(r) of range a. We choose the units $2m = \hbar = 1$ and denote the initial and final wave vectors by $\mathbf{k}_i$ and $\mathbf{k}_f$ respectively. The scattering wave function satisfies the Schroedinger equation.

$$[\nabla^2 - V(\mathbf{r}) + k^2]\psi(\mathbf{r}) = 0 \tag{1}$$

where $k = |\mathbf{k}_i| = |\mathbf{k}_f|$. The scattering amplitude, $f(\mathbf{k}_i, \mathbf{k}_f)$, is then defined as

$$f(\mathbf{k}_i.\mathbf{k}_f) = (\frac{-i}{4\pi}) \int d\mathbf{r} exp[-i\mathbf{k}_f.\mathbf{r}] V(\mathbf{r})\psi(\mathbf{r}) \tag{2}$$

where the integration is over the volume of the scatterer.

It is reasonable to assume that the potential will not severely disturb the motion of the projetile if

$$\frac{|V(\mathbf{r})|}{k^2} \ll 1 \qquad and \qquad ka \gg 1. \tag{3}$$

Thus $\psi(\mathbf{r})$ may be approximated as an incident plane wave modulated by some function which takes into account the small disturbance due to $V(\mathbf{r})$:

$$\psi(\mathbf{r}) = exp[i\mathbf{k}_i.\mathbf{r}]\phi(\mathbf{r}). \tag{4}$$

Substituting (4) into (3) we obtain

$$[\nabla^2 - V(\mathbf{r}) + 2\mathbf{k}_i.\nabla]\phi(\mathbf{r}) = 0\,. \tag{5}$$

The EA can be introduced in (5) by neglecting the ∇^2 term. The solution of the resulting equation, using the boundary condition $\phi(-\infty) = 1$, can be easily found to be :

$$\psi_{EA}(\mathbf{r}) = exp[ikz - (\frac{i}{2k}) \int_{-\infty}^{z} V(\mathbf{b}.z')dz']. \tag{6}$$

The subscript EA refers to the eikonal approximation.Aposteriori it can be checked that the validity domain of the above approximation is indeed given by (3).

Equation (6) shows that the effect of potential is to introduce only a change in phase of the incident wave. Substitution of (6) in (2), followed by a small angle scattering assumption, leads to the following expression for the scattering amplitude,

$$f(\mathbf{q})_{EA} = (\frac{ik}{2\pi}) \int d^2b exp[i\mathbf{q}.\mathbf{b}](exp[i\chi_O(\mathbf{b})] - 1) \tag{7}$$

where

$$\chi_O(\mathbf{b}) = (\frac{-1}{2k})\int_{-\infty}^{\infty} V(\mathbf{b}.z)dz.$$

It may be noted that no restriction has been placed on the parameter $|V|\frac{a}{k}$. In contrast the Born approximation and the WKB approximation are valid for $\frac{|V|}{k^2} \ll 1$; $\frac{|V|a}{k} \ll 1$ and $ka \gg 1$ and $\frac{|V|a}{k} \gg 1$, respectively.

3. The eikonal approximation in scalar optical scattering

The EA has been successfully adapted in light scattering problems by defining an effective potential $V(\mathbf{r}) = -k^2[n^2(\mathbf{r}) - 1]$. For a homogeneous sphere $n(\mathbf{r}) = n$ inside the sphere. Thus for small angle scattering the scattering function $S(\theta)$ assumes the form

$$S(\theta)_{EA} = \int_O^a bdbJ_O[2kbsin(\frac{\theta}{2})][1 - e^{-ik(n^2-1)\sqrt{(a^2-b^2)}}]. \tag{8}$$

$S(\theta)$ is related to $f(\theta)$ via the relation $S(\theta) = -ikf(\theta)$. For a completely absorbing sphere the scattering functin given by (8) reduces to Fraunhofer diffraction approximation. For forward scattering $S(\theta)_{EA}$ gives

$$S(0)_{EA} = x^2[\frac{1}{2} + i\frac{e^{i\rho}}{\rho} + \frac{(1 - e^{i\rho})}{\rho^2}] \tag{9}$$

where $\rho = ka(n^2-1)$ is the phase shift suffered by the cnetral ray and $x = ka$ is the size parameter or size of the particle in units of the wavelength of the scattering radiation. The total cross section or the extinction efficiency may then be obtained by using the optical theorem $Q_{ext} = \frac{4}{x^2}ReS(O)$.

For non-forward scattering a closed form analytic expression is not possible. Nevertheless, for a dielectric sphere $S(\theta)_{EA}$ can be expressed in terms of known functions by means of a series expansion for $\rho < 1$ as well as for $\rho > 1$ [3].

Expression for the scattering function in the EA has been obtained for the scattering of light by an infinitely long homogeneous circular cylinder for perpendicular as well as oblique incidence. For forward scattering the scattering function can be evaluated analytically for this problem too.

4. Validity domain of the eikonal approximation

By analogy with potential scattering the validity domain of the EA in optics may be expressed as

$$|n^2(\mathbf{r}) - 1| \ll 1 \qquad \textit{and} \qquad ka \gg 1. \tag{10}$$

There are however, three main differences between the situations of interest in potential scattering and the scattering of light by an obstacle. (i) In potential scattering, the EA has been examined in detail for scalar waves : the scattering of light requires a vectorial treatment. (ii) In potential scattering one is generally concerned with potentials that are energy independent : the effective potential in the scattering of light is energy dependent. (iii) In potential scattering a great deal of attention has been paid to the investigtion of the validity of the EA for continuously varying potentials. In light scattering problems, although one encounters obstacles with constant as well as variable refractive indices, there is always a cut-off at the boundary.

The effect of using a scalar approximation to vector scattering has been studied for homogeneous spheres and infinitely long cylinders for forward scattering. It has been found that for large particles ($ka \gg 1$), vectorial effects are indeed small. For $ka > 10.0$, these have been noted to be less than 3 percent [2].

The effect of energy dependence of the potential is that the EA can be looked upon as a high energy approximation only for a fixed $ka(n^2 - 1)$ [4]. This is in contrast to potential scattering where the EA can be viewed as a high energy approximation for fixed ka and also for fixed $\frac{|V|}{k^2}$.

For homogeneous particles, one does not achieve extra smoothness in the variation of refractive index by increasing ka. Thus for a given n increasing ka may not result in improvement of the EA. The numerical comparisons of exact and the EA forward intensities, indeed show that the EA does not improve consistently as ka increases for fixed n.

5. Relationship with the anomalous diffraction approximation

An approximation method extensively used for the analysis of light scattered by an obstacle is the anomalous diffraction approximation (ADA) [3]. Theoretically the validity domain of this approximation is also given by (10). This approximation can be obtained from the EA simply by replacing $(n^2 - 1)$ by $2(n - 1)$. In other words, straight line propagation is assumed here too, but instead of the phase shift $\chi_O(b) = ik(n^2 - 1)\sqrt{(a^2 - b^2)}$, the phase shift now is what one obtains in the geometrical optics approximation.

6. Corrections to the eikonal approximation

There have been many attempts to derive corrections to the EA [2,5-7]. Wallace [8] has developed a systematic expansion for infinitely often differentiable potentials. In this expansion the EA appears as the zeroth order term and the corrections systematically improve the EA results.

A treatment analogous to that of Wallace is not applicable in the context of light scattering because of cut-off at the boundaries. Therefore, the corrections to the EA have been defined by ignoring the possible discontinuous behaviour of the derivative of the refractive index at $|r| = a$. For a homogeneous sphere and a infinitely long homogeneous circular cylinder one obtains.

$$S(O)_{FCEA} = (1 + (\frac{\rho}{x}))S(O)_{EA} + (\frac{\rho x}{4})(exp(i\rho) - 1) \tag{11}$$

and

$$\begin{aligned} T(O)_{FCEA} &= T(O)_{EA} - (\frac{\rho^2}{4})[((\frac{4}{3})\rho_1 F_2(2, \frac{3}{2}, \frac{5}{2}; -\frac{\rho^2}{4}) - (\frac{\pi}{2})H_O(\rho)) \\ &\quad +(\frac{i\pi}{2})({}_1F_2(\frac{3}{2}, \frac{1}{2}, 2; \frac{\rho^2}{4}) - J_O(\rho))] \end{aligned} \tag{12}$$

where

$$T(O)_{EA} = (\frac{\pi x}{2})(H_1(\rho) + iJ_1(\rho)) \tag{13}$$

and H_O, H_1, J_O and J_1 are Struve and Bessel functions of zeroth and first order respectively, and ${}_1F_2$ is the generlised hypergeometric function.

7. Numerical comparisons

Numerical comparisons for forward scattering for spheres and for infinitely long cylinders show that, whilst the ADA is somewhat better than the EA if the entire ka domain is examined, it has been noted that the EA is superior to the ADA in the domain $x > 1.0; \rho < 4.0$. This conclusion is important since it relates to intermediate size particles and here the EA gives better results compared to the ADA. For particles with $x \gg 1$ or $x \ll 1$, simpler approximations, namely the Fraunhofer diffraction approximation and the RGDA, respectively may be used more profitably.

Typical percentage errors in the scattered intensity for spheres and infinitely long cylinders for $n = 1.05$ are shown in table 1. Here the percentage error is defined as $[i(0)_{ex} - i(0)_{appx}]x100/i(0)_{ex}$, where $i(0) = |S(0)|^2$ or $|T(0)|^2$ and subscript denote exact and approximate. It can be seen that the inclusion of corrections improves the results dramatically. The corrected results have been noted to be better than the ADA in the domain $x > 1.0$ and $\rho \leq 8.0$ [2,9].

	sphere			cylinder transverse electro-magnetic wave scattering		
x	EA	FCEA	ADA	EA	FCEA	ADA
1.0	-3.07	-8.43	1.88	2.05	2.05	6.77
3.0	1.36	-3.79	6.09	0.10	0.05	4.88
5.0	2.58	-2.56	7.20	0.11	0.01	4.83
10.0	3.67	-1.65	8.04	0.22	-0.17	4.69
20.0	4.90	-1.37	8.35	1.25	-0.37	4.59
30.0	6.60	-1.33	8.45	-	-	-
40.0	9.11	-1.31	8.55	-	-	-
60.0	16.65	-0.89	8.67	-	-	-
80.0	8.20	-0.67	6.80	-	-	-
100.0	2.98	-3.73	7.04	-	-	-

8. Applications

A measure of the ratio between the scattered intensities at a pair of convenient angles within the forward lobe is known to give a useful estimate of the size of the scattering particles. In view of the success of the FCEA in describing the scattered light intensities in the forward scattering lobe an estimate of the errors involved in using FCEA for fibre size determination was made for fibres in the diameter range $0.2\mu m$ to $4\mu m$ by Sharma and Somerford [10]. It was found that for the angle pair $(5^o, 2.5^o)$ the maximum errors for $n \leq 1.15$ were within 6%.

For rough particles the effect of the sphere roughness may be introduced by a Fermi function in phase $\chi(b)$. Such a procedure is physically based on the fact that that the refractive index decreases continuously in the region close to the surface and is not sharply cut-off [11] :

$$n(r) - 1 = \frac{(n-1)}{[1 + exp\{k(b-a)\}]} \tag{14}$$

The eikonal results for this model were found to be in good agreement with the measurements of the scattered intensity. Further improvements to this model have also been disucssed [8, 12]. Bourrely et.al. [13] have proposed a model of surface irregularities by a fractal type function. From an analysis based on the EA, they have been able to show that the behaviour of scattered intensity permits discrimination between smooth and rough surfaces and also between sizes of the irregularities.

If the wavelength of the incident raidation is greater than the Debye length, the plasma behaves like a dielectric medium and thus the EA has been used profitably for plasma density profiling in this context of the scattering of electromagnetic waves from a cylindircal plasma in the presence of an axial static magnetic field [14]. It has also been used in the study of the plasma heating by electromagnetic waves [15, 16].

There are many applications of light scattering where the ADA has found use. In most of these applications the EA can be used *mutatis mutandis* in place of the ADA by substituting $2(m-1)x$ by $(m^2-1)x$ with an advantage over the ADA in the domain $\rho < 4.0$ and $x > 1$.

Among other applications of the EA are its use in the (i) Transmission electron microscopy calculations for both image and diffraction compute simulations [17], (ii) Analysis of the scattering of very cold neutrons from hetrogenieties in a condensed medium to obtain their size and concentration [18]. (iii) Analysis of the backscattering of light [19], (iv) Analysis of diffraction of electromagnetic waves by periodic gratings [20], (v) Scattering of light by more than one sphere [21] and (vi) Light scattering by cladded optical fibres [22].

9. Conclusions and Discussions

In conclusion we note that although the validity of the EA has been studied in details and many applications made, further studies pertaining to its validity for particles of non-spherical shapes, and of attempts to extend the scalar analysis to vector scattering are very much desirable.

References

[1] Glauber R J 1959, Lectures in Theoretical Physics, Vol.I, Eds. W E Brittin and L G Dunham (Interscience : N Y) p. 315.

[2] Sharma S K and Somerford D J, 1990, Nuovo Cimento, D12, 719 and references therein.

[3] Van de Hulst H C, 1957, Light Scattering by Small Particles, (Wiley : N Y) Chap.11.

[4] Sharma S K, Roy T K and Somerford D J, 1988, J Mod Opt, 35, 1213.

[5] Alvarez-Estrada R F, Calvo M L andd Juncos del egido P, 1980, Optica Acta 27, 1367.

[6] Chen T W, 1989, Appl. Opt., 28, 4096.

[7] Perrin J M and Chiapetta P, 1985, Optica Acta, 32, 907.

[8] Wallace S J, 1973, Ann. Phys. (N Y), 78, 190.

[9] Sharma S K , 1993, Opt Commun, 100, 13.

[10] Sharma S K and Somerford D J, 1983, J Phys (GB); D17, 2191.

[11] Chiapetta P, 1979, Astron. Astrophys., 83, 348.

[12] Perrin J M and Lemy P L, 1983, Optica Acta, 30, 1223.

[13] Bourrely C, Chiapetta P and Torresani B, 1986, J Opt Soc Am., A3, 250.

[14] Sharma S K and Dasgupta B, 1987, Plasma Phys. Contr. Fusion, 29, 303.

[15] Chaudhery B J and Bhakhar B S, 1975, J Phys (GB), B8, L288.

[16] Gersten J I and Mittleman M H, 1975, Phys. Rev., A12, 1840.

[17] Gomez A and Castano V M, 1988, Phys. Status Solidi A107, 845.

[18] Stepnov A V and Shelagin A V, 1986, Sov. Phys. - Lebedev Instt. Rep. 3, 27.

[19] Sharma S K and Somerford D J, 1994, J Mod Opt, 41, 1433.

[20] Calvo M L and Juncos del Egido P, 1982, Optica Acta, 29, 1061.

[21] Chen T W, 1990, J Appl Phys, 67, 7147.

[22] Calvo M L and Juncos del Egido P, 1979, SPIE Proc., 213, 35.